Curve and Surface Reconstruction
From Regular and Non-Regular Point Sets

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Abstract

In this paper, we address the problem of curve and surface reconstruction from sets of points. We introduce regular interpolants which are polygonal approximations of planar curves and surfaces verifying a local sampling criterion. Properties of regular interpolants lead to new polygonal reconstruction methods from sets of organized and unorganized points. These methods do not need any parameter or additional information apart from the original points and allow unorganized sets of points to be easily handled.

1. Introduction

Computing a polygonal approximation of a curve or a surface from a set of unorganized sample points is a challenging problem in several fields such as computer vision, computer graphics and computational geometry. The concerned applications are, for instance, curve reconstruction in image analysis, three-dimensional model determination from laser range data and from computer vision processes (stereo-vision, reconstruction from apparent contours, . . .), reconstruction in medical imaging and the creation of computer models from existing parts in reverse engineering.

The problem faced in these applications can be stated in the following way: given a set of points of a plane curve or of a three-dimensional surface, construct a polygonal structure interpolating the sample points that reasonably captures the shape of the point set. In this paper, we introduce a new, simple and efficient algorithm for reconstructing curves and surfaces from unorganized sets of points.

1.1. Past work

Shape reconstruction from sets of points has been widely studied in the computer graphics and computational geometry communities in recent years. We focus here on the *Delaunay type* of reconstruction, in which local criteria allow to compute a mesh which interpolates the sample points and is included in the Delaunay triangulation of the point set. Some researchers directly build an interpolating mesh which is provably a subset of the Delaunay triangulation. Some others start by computing the Delaunay graph (for which very efficient algorithms exist) and then eliminate simplices which do not satisfy a given criterion.

An early reconstruction algorithm of the latter kind is the Delaunay tetrahedrization "sculpting" heuristic of Boissonnat [4]: tetrahedra are progressively eliminated according to a geometric criterion taking into account the areas of their faces. In [13], Veltkamp computes γ-graphs in a similar way, though he eliminates tetrahedra using a slightly different criterion.

The α-shapes of Edelsbrunner and Mücke [6] have also been used for surface reconstruction. The α-shapes represent a finite set of points at different levels of details. Each member of the parameterized family is a simplicial complex which is a subset of the Delaunay triangulation. The spectrum of the α-shapes gives an idea of the overall shape and dimensionality of the point set. In [6], experiments are made using α-shapes for reconstruction. Starting with the Delaunay graph, simplices are eliminated according to the diameter of their circumscribed ball. The main difficulty appears to be that the parameter of the α-shape must be chosen experimentally. Also, in many cases, variations of sampling granularity imply that there is no ideal value.

Recently, a number of theoretical results have been obtained on various Delaunay-based approaches [1, 2]. In particular, emphasis has been put on the definition of "good" samplings of a shape and on the development of algorithms with provable guarantees, i.e. such that the reconstruction is guaranteed to be topologically correct and convergent to the original surface as the sampling granularity increases. Amenta *et al.* [1] have introduced such an algorithm, based on Voronoi diagrams and Delaunay triangulations, which constructs an interpolating shape called the *crust*. Their definition of a "good" sampling captures the intuitive notion that featureless areas can be reconstructed from fewer samples. Even though they overcome some of the drawbacks of α-shape algorithms, the methods proposed by Attali and Amenta *et al.* work only well in 2D. In the 3D case, reconstruction from "good" samplings is much more complex and heuristics have to be applied.

1.2. This paper

In this paper, we introduce *regular interpolants* which are polygonal approximations verifying a new sampling criterion. By studying local properties of regular interpolants, we derive reconstruction algorithms for shapes that are regularly sampled, or, in other words, such that the polygonal approximation that connects adjacent points on the original shape is a regular interpolant. By contrast to the approaches presented above, no parameter or additional information is needed. Similar algorithms can also be applied to sets of points which do not define regular interpolants. The idea then is to iteratively determine locally regular configurations of simplices. In that case, more simplices will be computed than is really needed, revealing the internal structure of the data, and a further step based on heuristics is required to extract interesting parts from this structure. We have applied this method to different types of data, in particular real data coming from computer-vision reconstruction process, and the results obtained show the correctness of the approach.

The paper is organized as follows. Section 2 introduces definitions and geometric structures that are used in the rest of the paper. Regular interpolants of point sets of $\mathbb{R}^2$ are examined in Section 3 and a reconstruction method is proposed. Section 4 then moves up to the 3D case. And Section 5 shows that the properties of regular interpolants can be turned into heuristics for reconstructing non-regular point sets. Several significant examples and results are given, before concluding. Note that only the main results are stated in this paper. The details of the proofs and additional properties of regular interpolants are left for a future paper.

2. Definitions

In this section, we recall and introduce notions that are used in the rest of the paper to describe the shape of an object. Distances are supposed to be Euclidean throughout the paper.

Let $\mathcal{P}$ be a set of n points of $\mathbb{R}^m$, $m = 2$ or 3, called *sample points*. We first define what we mean by a piecewise-linear curve interpolating the points of $\mathcal{P}$:

Definition 1 *A* piecewise-linear interpolant *(or simply interpolant for short) of $\mathcal{P}$ is a $(m-1)$-simplicial complex $\mathcal{O}$ the 0-skeleton of which is precisely $\mathcal{P}$.*

An interpolant is thus a set of edges in 2D, and a set of triangles in 3D, interpolating all the sample points in $\mathcal{P}$. From now on, let $\mathcal{O}$ be a *closed* interpolant of $\mathcal{P}$, i.e., an interpolant such that each point of $\mathcal{P}$ is shared by at least two different edges and such that each edge in 3D is shared by at least two different triangles. To define what we mean by a "regular" interpolant, we need to introduce the notions of local thickness and local granularity. Before that, we define the *neighborhood* $\mathcal{N}_\mathcal{O}(P)$ of a sample point P on $\mathcal{O}$ as the set of points of $\mathcal{O}$ having P as their closest sample point.

Definition 2 *The* local granularity $g_\mathcal{O}(P)$ *of $\mathcal{O}$ at P is the minimum real positive number d such that for all points M of $\mathcal{N}_\mathcal{O}(P)$, the open ball $B(M, d)$ contains the point P.*

The local granularity at P can be seen as the granularity of $\mathcal{O}$ (as defined for instance in [12]) over the neighborhood $\mathcal{N}_\mathcal{O}(P)$.

Next, we define the notion of adjacency of points of $\mathcal{P}$ on $\mathcal{O}$:

Definition 3 *k distinct points, $2 \leq k \leq m$ of $\mathcal{P}$ are said to be* adjacent *on $\mathcal{O}$ iff they are cosimplicial (i.e. if they are vertices of the same $(k-1)$-simplex of $\mathcal{O}$).*

Now, we introduce a "discrete" version of the medial axis, usually defined as the locus of centers of closed balls that are maximal with respect to inclusion.

Definition 4 *The* medial axis *of $\mathcal{O}$ is the locus of points of $\mathbb{R}^m$ which are closer to two or more non-cosimplicial sample points than to any other point of $\mathcal{P}$.*

Thus, the medial axis is the locus of points which are closer to non-adjacent points.

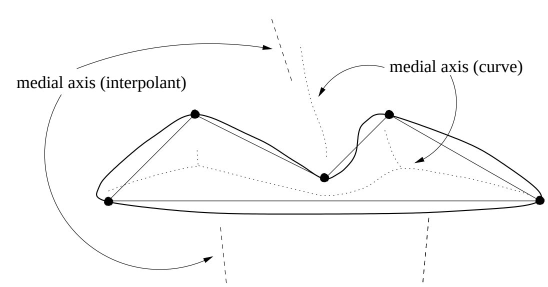

Figure 1. A curve, an interpolant of five sample points on the curve, the medial axis of the curve (dotted line) and the medial axis of the interpolant (dashed line).

In the case of a continuous curve C, the medial axis has several properties:

- each component of $\mathbb{R}^2 \setminus C$ contains a part of the medial axis,
- by definition, the medial axis nowhere intersects C,
- each point of C contributes to the medial axis, in the sense that it lies on the boundary of at least one maximal ball included in $\mathbb{R}^2 \setminus C$.

Consider now Fig. 1, where we have displayed a continuous closed curve C, an interpolant of five sample points of C and the medial axis of both curves. This figure shows that if the sampling of a curve is not "dense enough", then we may lose components of the medial axis by going discrete. It may also be that the medial axis intersects the interpolant. We will take care of these situations in the following definitions.

In the continuous case, the thickness of a shape is defined as the minimum distance from a point of the shape to its medial axis. We here introduce a local version of this definition, taking into account the peculiarities of the medial axis of polyhedral shapes as we defined it.

Definition 5 *The* local thickness $e_{\mathcal{O}}(P)$ *of* $\mathcal{O}$ *at the point* $P \in \mathcal{P}$ *is either:*

- *0 if P does not contribute to the medial axis in each of the components of $\mathbb{R}^m \setminus \mathcal{O}$ of which it is a vertex or if any of these contributions intersects $\mathcal{O}$,*
- *the minimum distance from P to the medial axis of $\mathcal{O}$ otherwise.*

Now that granularity and thickness have been introduced, we can define what we mean by a "good" interpolant.

Definition 6 *$\mathcal{O}$ is said to be a* regular interpolant *of $\mathcal{P}$ if at each point P of $\mathcal{P}$ the local granularity $g_{\mathcal{O}}(P)$ is strictly smaller than the local thickness $e_{\mathcal{O}}(P)$. The point set $\mathcal{P}$ is called* regular *if it admits at least one regular interpolant.*

In [1], local sampling conditions are also introduced. However, these conditions apply to the smooth curves or surfaces from which the sample points have been drawn. They focus more on the shape variations of the original curve or surface rather than on the connections between adjacent points. But the fact that these connections can be recovered depends much more on the properties of the interpolating shape than on the properties of the original continuous shape.

This is the issue that this paper examines. It turns out that focusing on sample points and their interpolants leads to conditions that are less restrictive on the sampling and thus is more efficient to handle unorganized sets of points.

3. Regular interpolants of 2D point sets

We now focus on closed regular interpolants of point sets of $\mathbb{R}^2$. The regularity condition allows us to derive local properties at sample points. Given a point set corresponding to sample points of a closed regular interpolant, these properties can in turn be used to recover the associated interpolant.

3.1. Properties

The fundamental property of adjacent points is that they define an empty ball:

Property 3.1 *Let P_i and P_j be two adjacent points on $\mathcal{O}$. Then the ball B_{ij} with diameter P_iP_j is void of sample points other than P_i and P_j.*

A first consequence of this property is that an edge P_iP_j of a regular interpolant belongs to the Gabriel graph (GG)of the point set $\mathcal{P}$ and thus to the Delaunay graph of $\mathcal{P}$. (In the GG, two vertices P_i and P_j are connected if the ball centered at $(P_i + P_j)/2$ and passing through P_i and P_j contains no other point in its interior.) However, edges of a regular interpolant do not necessarily belong to the relative neighborhood graph (RNG) of the point set, which is itself included in the Gabriel graph. (In the RNG, two vertices P_i and P_j are connected if the associated lune – the intersection of the two closed balls with centers P_i and P_j and radius $d(P_i, P_j)$ – is empty of other points.)

Now, not all edges of the Gabriel graph of $\mathcal{P}$ correspond to edges of a regular interpolant. The following property allows to discriminate:

Property 3.2 *The sampling points adjacent to $P_i \in \mathcal{O}$, where $\mathcal{O}$ is a regular interpolant of $\mathcal{P}$, are the points $P_j, P_k \in \mathcal{P}, P_j \neq P_k$ such that:*

1. *P_iP_j and P_iP_k are edges of the Gabriel graph;*
2. *the local granularity at P_i is minimal (i.e., among the neighbors of P_i in the Gabriel graph, P_j and P_k are the closest to P_i).*

3.2. Reconstruction

Properties of regular interpolants define a theoretical context in which reconstruction of polygonal curves from their sampling points can be easily achieved. The simplest approach for reconstructing a regular interpolant (when it exists) consists in applying Property 3.2 directly. For each point P_i of $\mathcal{P}$, its closest point P_j in $\mathcal{P}$ is determined, giving one edge of the polygonal approximation. Then, by Property 3.2, P_i is linked to its second adjacent point P_k.

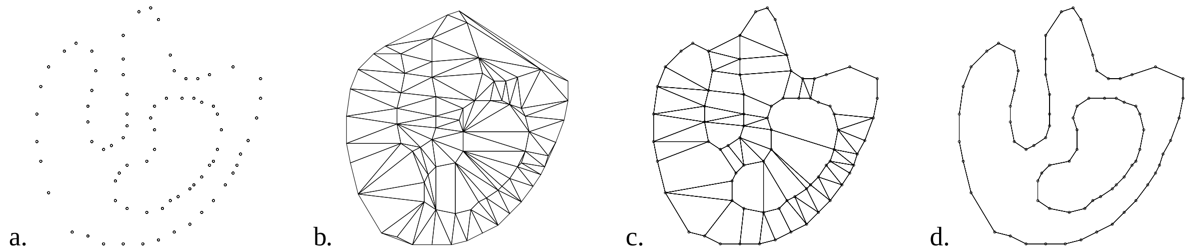

Figure 2. Example of polygonal reconstruction from a regular sampling of a plane curve. a. Sample points. b. The Delaunay graph. c. The Gabriel graph. d. The regular interpolant.

Algorithm 1 2D reconstruction, regular case

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for all point P_i ∈ 𝒫: do
    find the closest point P_j ∈ 𝒫 to P_i
    find the point: S = min_{M∈𝒫'} d(M, P_i), with 𝒫' the
        subset of 𝒫 of points P_k / P_j ∉ B_ik
    add P_iS and P_jS to the polygonal reconstruction
end for
```

An example of reconstruction is shown in Figure 2, along with the Delaunay and Gabriel graphs of the point set.

A few remarks are in order:

1. The algorithm uses the fact that the point S which is determined defines necessarily, with the point P_i, an edge of the Gabriel graph of $\mathcal{P}$. This is not true if $\mathcal{P}$ does not come from a regular interpolant. In that case, the fact that each edge belongs to the GG must be checked.

2. In its simplest form, the algorithm takes $O(n^2)$ time, where n is the number of sample points. This complexity can easily be reduced to $O(n \log n)$ time by choosing appropriate nearest neighbor functions (for nearest neighbor searching, see for example [3, 11]).

We can also proceed by first computing one of the geometric graphs mentioned and then eliminating the superfluous edges. For example, the Gabriel graph of $\mathcal{P}$ can be computed. Such an operation can be accomplished in $O(n \log n)$ time by use of the Delaunay triangulation [9]. Then, eliminating from this graph all edges that do not verify the second condition of Property 3.2 will give the result. Since the expected time for the elimination step is linear, this leads also to an $O(n \log n)$ algorithm.

4. Regular interpolants of 3D point sets

What happens in the three-dimensional case ? We will see in this section that, in a manner similar to the planar case, properties of regular interpolants can be used to perform surface reconstruction. As in the 2D case, we assume that $\mathcal{O}$ is a closed regular interpolant of a point set $\mathcal{P}$.

4.1. Properties

As in 2D, the basic property concerns the emptiness of a ball whose boundary goes through sample points:

Property 4.1 *Let P_i, P_j and P_k be three adjacent points on $\mathcal{O}$. Then the ball B_{ijk} with center and radius those of the circle circumscribed to P_i, P_j and P_k is void of sample points other than P_i, P_j and P_k.*

A consequence of this property is that all triangles of a regular interpolant belong to a graph which is an extension of the Gabriel graph to 3D. Even though this extension (from edges in 2D to triangles in 3D) seems pretty straightforward, it does not seem to have been studied in the literature. Lacking of an appropriate name, we simply call it the 3D Gabriel graph (3DGG). The similitude with the planar case goes further. Indeed, the 3DGG is itself included in the Delaunay graph of $\mathcal{P}$.

By Property 4.1, triangles of $\mathcal{O}$ belong to the 3DGG. However, not all simplices of the 3DGG define triangles of $\mathcal{O}$. An argument similar to the 2D case shows that:

Property 4.2 *Let P_i, P_j be two adjacent points of a regular interpolant $\mathcal{O}$ of $\mathcal{P}$. The sample points P_k and P_l forming adjacent triples with P_i and P_j are those points of $\mathcal{P}$ such that:*

1. *the triangles $P_iP_jP_k$ and $P_iP_jP_l$ correspond to edges of the 3DGG;*

2. *the local granularities at P_i and P_j are minimal (i.e., points P_k and P_l are those points of the 3DGG which minimize the granularities at P_i and P_j).*

4.2. Reconstruction

Based on the above properties, the algorithm for reconstructing the interpolant of a regular sample point set is very similar to the planar case. The idea is to stick to Property 4.2. Let $E(P_i, P_j, P_k)$ denote the edge (P_i, P_j) associated to the triangle (P_i, P_j, P_k). Algorithm 2 is used to reconstruct the interpolant.

Algorithm 2 3D reconstruction, regular case

1: **find** an initial adjacent triple P_0, P_1, P_2
2: **initialize** the list *Ledges* with the three edges $E(P_0, P_1, P_2)$, $E(P_1, P_2, P_0)$ and $E(P_0, P_2, P_1)$
3: **for all** edges $E(P_i, P_j, P_k)$ in *Ledges* **do**
4:   **find** the point S: $S = \max_{M \in \mathcal{P}'} \widehat{P_i, M, P_j}$, with $\mathcal{P}'$ the subset of $\mathcal{P}$ of points P_l / $P_k \notin B_{ijl}$
5:   **add** (S, P_i, P_j) to the triangular mesh
6:   **if** $E(P_i, S, P_j)$ and $E(P_j, S, P_i)$ are not in *Ledges* **then**
7:     **add** them to the list
8:   **else**
9:     **remove** them
10:   **end if**
11:   **remove** $E(P_i, P_j, P_k)$ from *Ledges*
12: **end for**

Note the following:

1. To initialize *Ledges*, we can proceed as follows: choose a point P_0 in $\mathcal{P}$, find its nearest neighbor P_1 and the point P_2 maximizing the angle $\widehat{P_0, P_2, P_1}$.

2. If $\mathcal{O}$ is not connected, i.e., is composed of more than one component, then Algorithm 2 must be iterated.

3. The algorithm uses the fact that if S maximizes the angle $\widehat{P_i, M, P_j}$ then it minimizes also the local granularity at $P_i P_j$. Indeed, if S minimizes the local granularity at $P_i P_j$ but does not maximize $\widehat{P_i, M, P_j}$, then it means that the ball circumscribed to P_i, P_j and S contains another point and thus, that $P_i P_j S$ does not belong to the 3DGG.

4. If n is the number of sampling points then the number of triangles is linear in n and thus the complexity of the above algorithm in its simplest form is $O(n^2)$. Optimizations can be made by using appropriate search functions.

As in the 2D case, the resulting triangular mesh is included in the 3DGG of $\mathcal{P}$. Therefore, another class of approaches consists in eliminating triangles from the Delaunay graph of $\mathcal{P}$ according to Property 4.2. The complexity in that case would be the complexity of computing the Delaunay triangulation, i.e. $O(n^2)$ [10]. The interest of Algorithms 1 and 2 is that they determine valid simplices directly instead of eliminating simplices from a pre-computed graph. This will be useful in the case of unorganized sets of points as shall be seen in the next section.

5. Unorganized sets of points

In the previous sections, we have seen how to reconstruct the interpolant of a point set when the sampling has some regularity property. What can now be recovered using the techniques above when the point set does not come from a regular interpolant? We address this problem in this section and we propose an approach that leads to polygonal structures which describe well the adjacency between points. Such structures can then be used to extract interpolating curves or surfaces which are manifold.

5.1. Minimal interpolants

With Properties 3.2 (in 2D) and 4.2 (in 3D), we have seen a simple criterion for determining simplices to obtain the interpolant of a regular point set. The idea was to compute, from one simplex, adjacent simplices with minimal granularities such that the two simplices are locally regular (i.e., the circumscribed ball to one simplex does not contain the adjacent simplex). Those simplices were necessarily in the Gabriel graph.

When the point set is not regular, we want the interpolant to have similar properties. However, in this case, the adjacent simplex with minimal granularities such that the two simplices are locally regular does not necessarily belong to the Gabriel graph. One assumption that we make is that the interpolant must be a subgraph of the Gabriel graph. Accordingly, we use the following conditions to iteratively build the interpolant $\mathcal{O}$:

1. the $(m-1)$-simplices S_2 adjacent to an already computed $(m-1)$-simplex S_1 on $\mathcal{O}$ are such that: their granularities are minimal and the ball circumscribed to S_2 does not contain S_1.

2. the $(m-1)$-simplices of $\mathcal{O}$ belong to the Gabriel graph of $\mathcal{P}$.

These two conditions are equivalent to Properties 3.2 and 4.2 in the case of regular interpolants (i.e., simplices are those in the Gabriel graph with minimal granularities), but they are more restrictive in the general case. It is a heuristic which, when applied, eliminates some of the potential adjacent points of a point in 2D or an edge in 3D. The resulting approximation corresponds to a subgraph of the Gabriel graph of $\mathcal{P}$ which we call a *minimal interpolant* of $\mathcal{P}$. It should be noted that if the minimal interpolant is a regular interpolant when the point set is regular (the sampling of a closed regular interpolant), it does not necessarily correspond to a regular interpolant in the general case.

The algorithms we derive from these conditions are very similar to the regular case. For instance, in 3D, and keeping the same notations as before, we use Algorithm 3.

Algorithm 3 3D reconstruction, non-regular case

find an initial adjacent triple P_0, P_1, P_2
initialize the list *Ledges* with the three edges $E(P_0, P_1, P_2)$, $E(P_1, P_2, P_0)$ and $E(P_0, P_2, P_1)$
for all edges $E(P_i, P_j, P_k)$ in *Ledges* **do**
  find the point S: $S = \max_{M \in \mathcal{P}'} \widehat{P_i, M, P_j}$, with $\mathcal{P}'$ the subset of $\mathcal{P}$ of points $P_l \ / \ P_k \notin B_{ijl}$
  if (S, P_i, P_j) is in the Gabriel graph **then**
    if (S, P_i, P_j) does not already exist **then**
      add it to the triangular mesh
      add $E(P_i, S, P_j)$ and $E(P_j, S, P_i)$ to *Ledges*
    end if
  end if
  remove $E(P_i, P_j, P_k)$ from *Ledges*
end for

The algorithm complexity is $O(Nn)$ where N is the number of triangles. N depends on the natural organization of $\mathcal{P}$. If $\mathcal{P}$ describes a surface then N is linear in n. But in the worst case, $\mathcal{P}$ can describe a volume and N may be $O(n^2)$. Optimizations can also be made by using appropriate search functions.

5.2. Manifold interpolants

In the case of a non-regular point set, the minimal interpolant contains more triangular faces than needed. In particular, minimal interpolants are generally non-manifold, which is undesirable. One way to proceed is to extract a manifold surface by considering simplices that are simply connected, i.e. edges having each of their endpoints shared by a unique edge in 2D and triangles having each of their edges shared by a unique triangle in 3D. We start by grouping simply connected simplices. The idea is then to iteratively eliminate groups with small numbers of simplices and recompute groups until there is only one component.

5.3. Results

We show some results of the application of the above algorithms to non-regular point sets. In 2D, Fig. 3 shows the polygonal approximation computed from sample points on a curve and the minimal interpolant of a set of random points in the plane (Fig. 3-(d)). Note how the adjacency relations computed reveal the natural organization of the point set. Note also differences with the crust [1] of $\mathcal{P}$ (Fig. 3-(c)) which does not produce an interpolant in that case.

In 3D, Fig. 4 shows the triangular mesh obtained when reconstructing the interpolant of a set of randomly distributed points on a torus. Note that by contrast to the case of points randomly distributed on the sphere, points on a torus do not, in general, have a regular interpolant.

Fig. 5 shows an example using data coming from reconstruction from image sequences. For this example, we have extracted a manifold surface from the output of the minimal Gabriel graph, as explained above.

6. Conclusion

In this paper, we have introduced the notion of regular interpolant of a 2D or 3D point set. From the properties of regular interpolants, reconstruction methods have been proposed for regular and non-regular sets of points. In the case of non-regular samplings, the reconstructed shapes are not necessarily manifold and we apply heuristics to extract manifold parts. The resulting reconstruction methods lead to very simple and efficient algorithms that iteratively build interpolants of sets of points.

We are currently working on further improvements and applications of our method. First, the final step is, in the current version, simple and consists in extracting connected components. This part of the process can be improved by considering other heuristics. We want for instance to be able to achieve such tasks as filling holes in the surface. Second, we are currently investigating more closely what it means for a point set to be regular and how to ensure that a point set will have a regular interpolant. This is a crucial issue for applications. We are also investigating further theoretical implications of regular interpolants.

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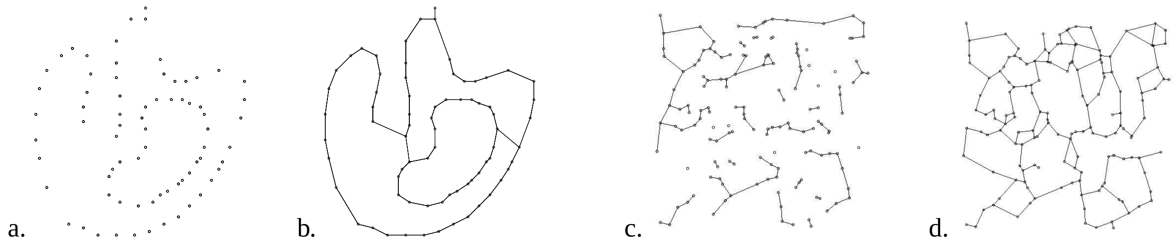

Figure 3. Interpolants of non-regular planar point sets. a. Sample points on a curve. b. The computed interpolant. c. The crust [1] of a set $\mathcal{P}$ of randomly distributed points in the plane. d. The computed interpolant of $\mathcal{P}$.

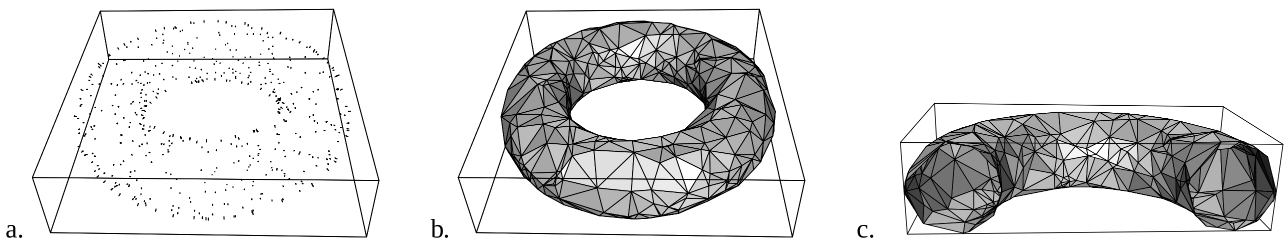

Figure 4. a. 500 points randomly distributed on a torus. b. The manifold interpolant. c. A section of the interpolant.

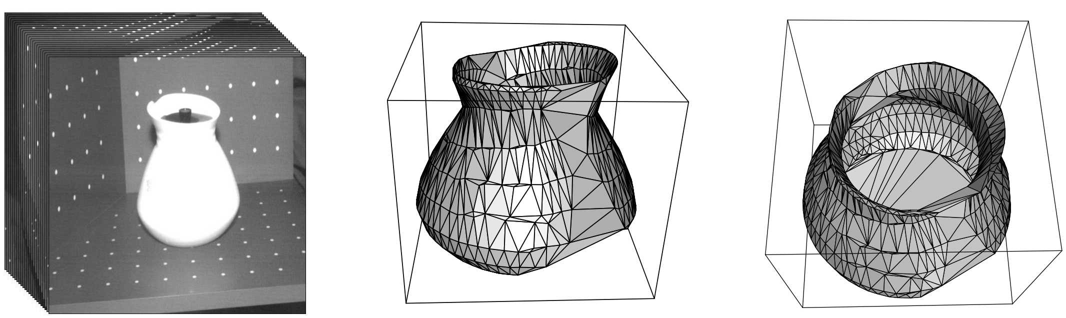

Figure 5. Examples of manifold surfaces. Data coming from reconstruction from image sequences [5].

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